11. Mapas Simpléticos

PGF 5005 - Mecânica Clássica

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(Referências principais: Reichl, *The Transition to Chaos*,1992; Lichtenberg e Lieberman, *Regular and Chaotic Motion*, 1992)

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Mapas Simpléticos

- Sistemas discretizados, conservativos, descritos por relações de recorrência.
- Exemplos de mapas bidimensionais (com duas variáveis): Mapa Padrão, Mapa de Hénon.

- Simulam mapas de Poincaré, de sistemas com Hamiltonianas quase integráveis, com dois graus de liberdade e uma constante de movimento.
- Sistemas mixtos: órbitas regulares e caóticas.
- Parâmetros de controle.
- Pontos fixos, estabilidade.

I – Mapas Simpléticos

(Baseado no Capítulo 3 do livro *Regular and Chaotic Motion*, Lichtenberg/Lieberman)

Mapeamento do Sistema Hamiltoniano

Sistema Integrável

$$H(J_1, J_2) = E$$

E: cte. de movimento

$$J_1 = J_1^0$$

$$J_2 = J_2^0$$

$$\theta_1 = \theta_1^0 + \omega_1 t$$

$$\theta_2 = \theta_2^0 + \omega_2 t$$

$$\alpha \equiv \frac{\omega_1}{\omega_2}$$

$$\frac{\omega_1}{\omega_2} = \frac{\Delta\vartheta_1/\Delta t}{\Delta\vartheta_2/\Delta t}$$

Se $\alpha = \frac{s}{r}$ r,s inteiros (primos)

$\Rightarrow$ órbitas periódicas, s(r) voltas em θ_1 (θ_2)

Se $\alpha \neq \frac{s}{r}$

$\Rightarrow$ órbitas quase-periódicas

r = 1

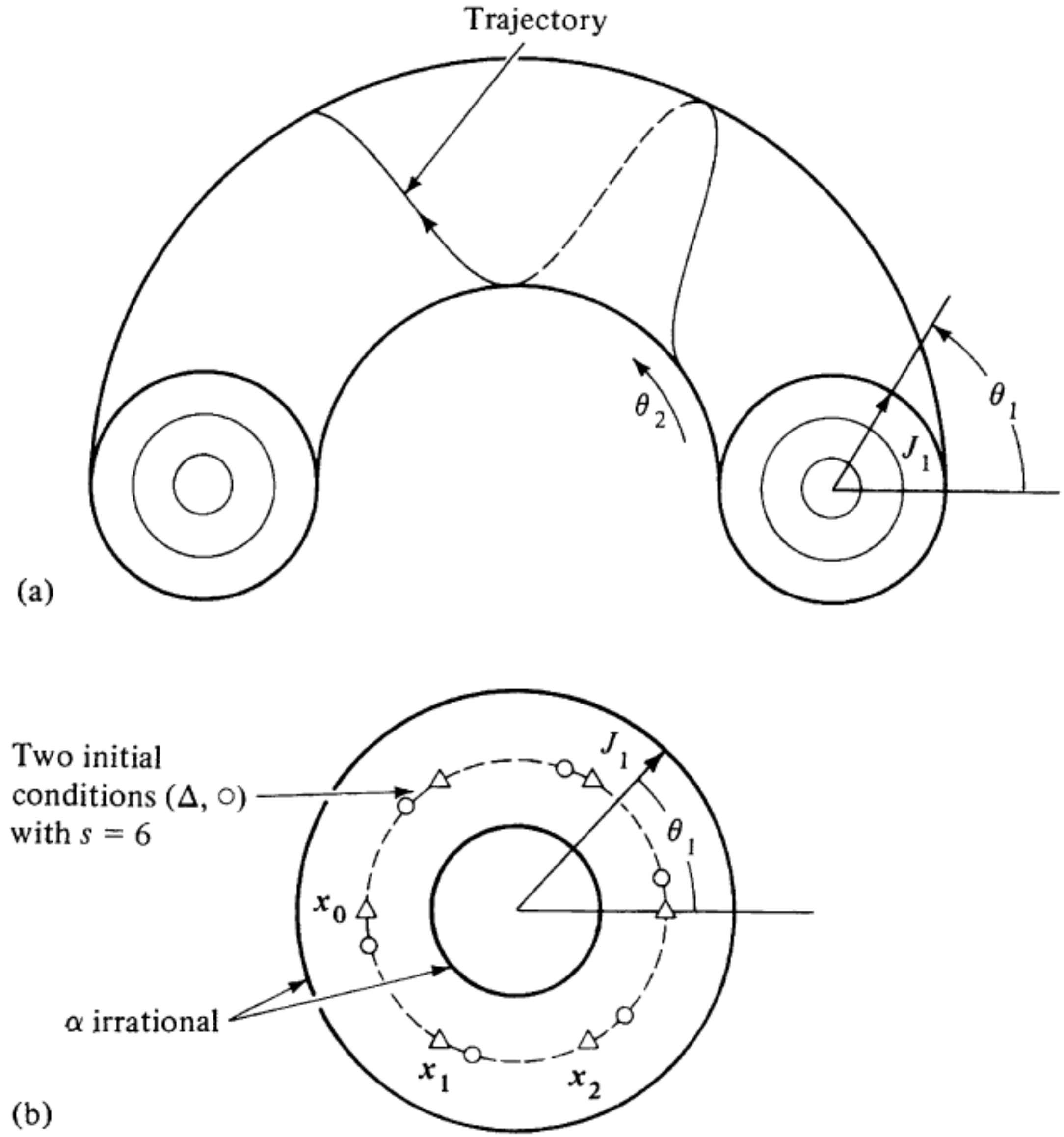


Figure 3.1. Motion of a phase space point for an integrable system with two degrees of freedom. (a) The motion lies on a torus $J_1 = \text{const.}$, $J_2 = \text{const.}$ (b) Illustrating trajectory intersections with a surface of section $\theta_2 = \text{const.}$ after a large number of such intersections.

Mapa de Poincaré
Sistema integrável

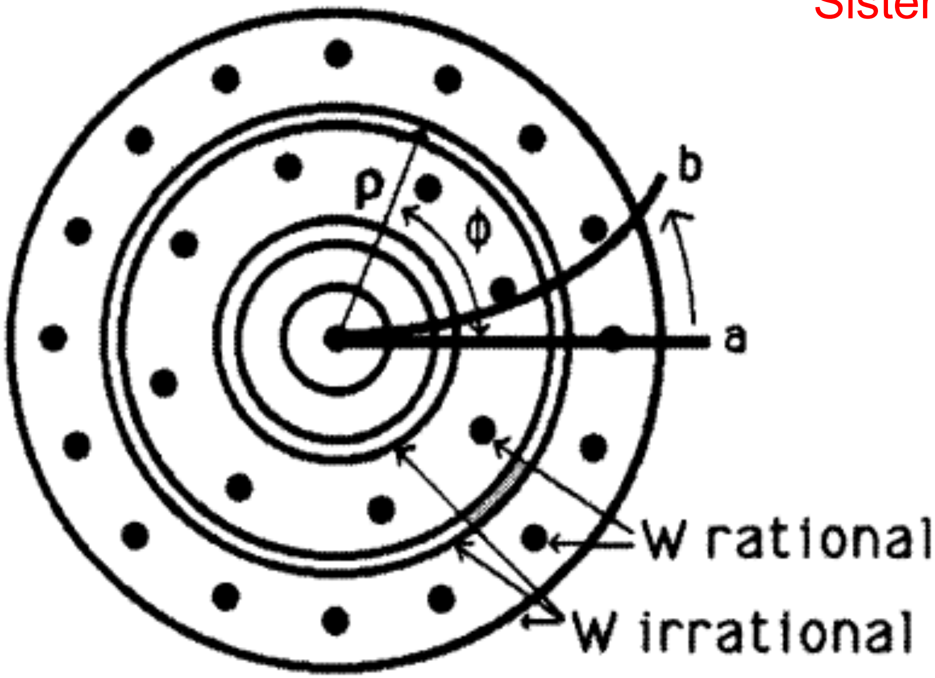
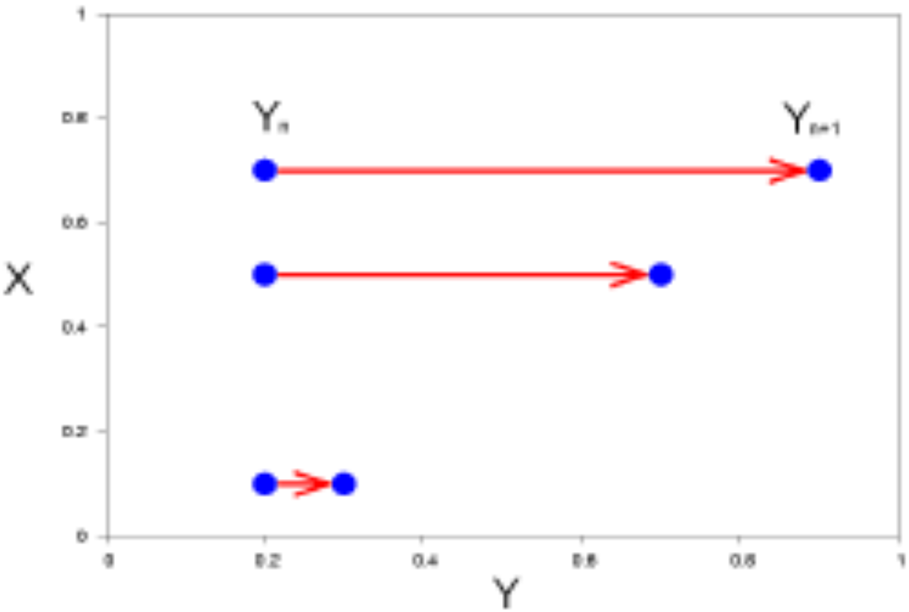


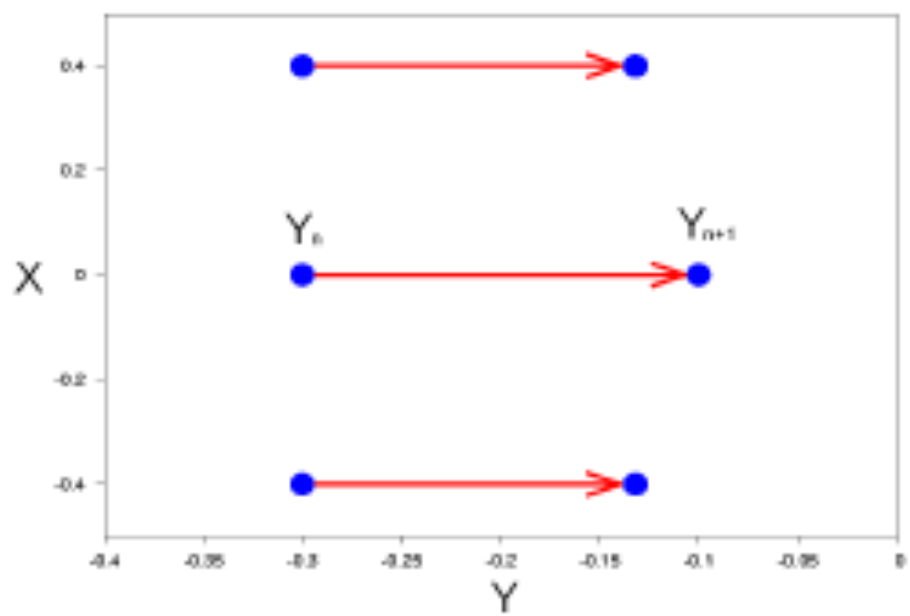
Figure 3.2.1. For integrable systems, the twist map consists of trajectories that densely fill a circle (irrational winding number w) and discrete, periodic points (rational winding number w). The rate at which a trajectory completes one revolution of the circle depends on the radius. Thus an initial line of points, a, becomes twisted, b, by the map.

Evolução de Y

Twist



Não-*twist*



Como pode ser notado, a evolução para Y é diferente de acordo com o tipo de mapa, *twist* ou não-*twist*, sendo monotonicamente crescente para o primeiro e não-monotônica para o segundo.

Mapas Twist

Mapa de Poincaré: Intersecções das trajetórias no plano $J_1 \times \theta_1$ (θ_2 = cte., $J_2 > 0$)

Duas intersecções sucesivas $\Rightarrow$

$$\Delta t = \frac{2\pi}{\omega_2} \qquad \Delta\theta_1 = \omega_1 \Delta t = 2\pi \frac{\omega_1}{\omega_2} = 2\pi\,\alpha$$

$$\alpha = \alpha(J_1)$$

Mapa integrável

$$J_{n+1} = J_n,$$

$$\theta_{n+1} = \theta_n + 2\pi\alpha(J_{n+1})$$

Omitindo o índice 1, temos uma sequência de valores discretos de $J_n = J$ e θ_n que varía com n

Mapas (twist e não twist) são conservativos

$$J_{n+1} = J_n,$$

$$\theta_{n+1} = \theta_n + 2\pi\alpha(J_{n+1})$$

$$\frac{\partial(J_{n+1}, \theta_{n+1})}{\partial(J_n, \theta_n)} \equiv [\theta_{n+1}, J_{n+1}] = 1$$

Se $J_n = J$, $\alpha(J)$ é constante e o mapa é integrável

Mapa Canônico para Sistema Hamiltoniano

Sistema Quase Integrável $$H(\boldsymbol{J}, \boldsymbol{\theta}) = H_0(\boldsymbol{J}) + \epsilon H_1(\boldsymbol{J}, \boldsymbol{\theta})$$

Amplitude da perturbação $\varepsilon \sim 0$

Mapa de Poincaré: Intersecções das trajetórias no plano $J_1 \times \theta_1$ (θ_2 = cte., $J_2 > 0$)

Mapa quase integrável

$$J_{n+1} = J_n + \epsilon f(J_{n+1}, \theta_n),$$

Omitindo o índice 1 de J_1

$$\theta_{n+1} = \theta_n + 2\pi\alpha(J_{n+1}) + \epsilon g(J_{n+1}, \theta_n)$$

Funções f, g periódicas em θ

For many interesting mappings, f is independent of J, and $g \equiv 0$. Then (3.1.13) takes the form of a *radial twist mapping*:

$$J_{n+1} = J_n + \epsilon f(\theta_n), \tag{3.1.17a}$$

$$\theta_{n+1} = \theta_n + 2\pi\alpha(J_{n+1}). \tag{3.1.17b}$$

II - Mapa Padrão

(Baseado no Capítulo 3 do livro

The Transition to Chaos, L. E. Reichl, 1992)

L. E. Reichl, Physica Scripta. Vol. T39, 90-95, 1991.
The Quantum and Stochastic Manifestations of Chaos

$$p_{n+1} = p_n - \frac{k\ sen(2\pi x_n)}{2\pi}$$

$$x_{n+1} = x_n + p_{n+1}$$

$$\begin{pmatrix} p_{n+1} \\ x_{n+1} \end{pmatrix} = T_K \begin{pmatrix} p_n \\ x_n \end{pmatrix} = \begin{pmatrix} p_n - \frac{K}{2\pi}\sin(2\pi x_n) \\ x_n + p_{n+1} \end{pmatrix}$$

Let us first consider the standard map for the case when $K = 0$

$$\begin{pmatrix} p_{n+1} \\ x_{n+1} \end{pmatrix} = T_0 \begin{pmatrix} p_n \\ x_n \end{pmatrix} = \begin{pmatrix} p_n \\ x_n + p_{n+1} \end{pmatrix} \qquad p_n = p_0 \text{ for all } n$$

When $p_0 = \frac{N}{M}$ the orbit will be periodic with period M

after M steps the orbit repeats itself (mod 1), $x_M = x_0 + N = x_0 \pmod 1$

If p_0 is irrational, then x_n never repeats itself

For $K = 0$, the winding number $w(p_0) = p_0$

$$\omega\,(p_0) = x_{n+1} - x_n = \Delta x = p_o$$

Órbitas periódicas e quase-periódicas

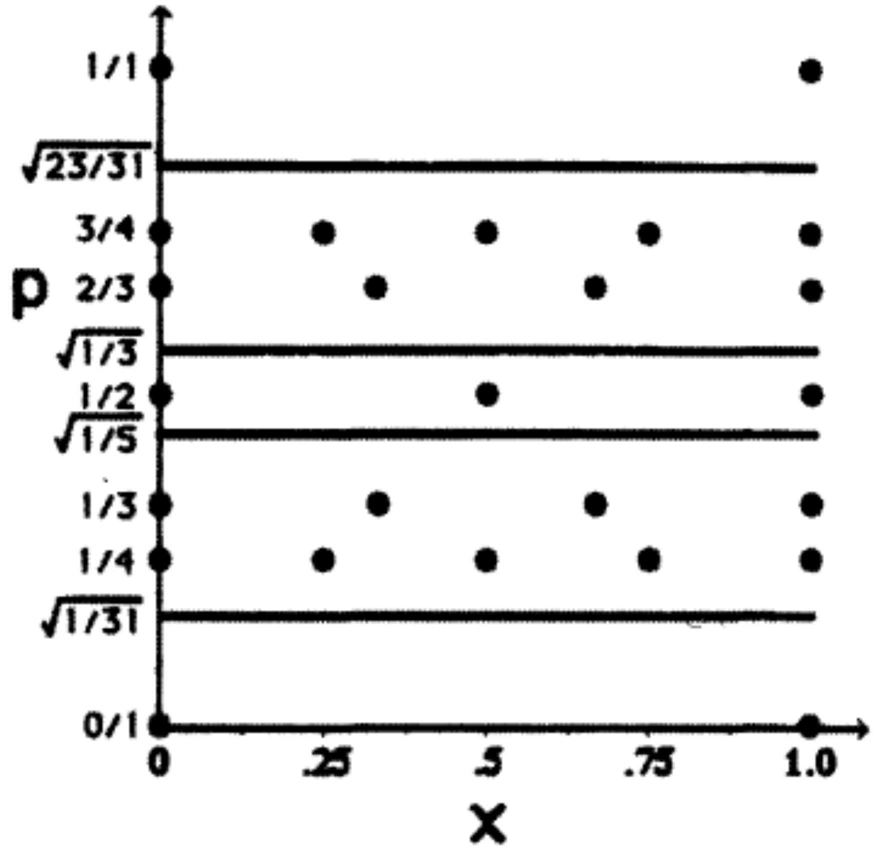


Figure 3.5.1. The behavior of some typical orbits of the integrable twist map Eq. (3.5.1). For this case, the winding number $w = p$. Orbits with irrational winding number fill a line densely, while those with rational winding number form a discrete set of periodic points.

For the case $K \neq 0$, the winding number can be defined as

$$w(p_0) = \lim_{n \to \infty} \frac{x_n - x_0}{n}$$

and can be used to characterize both periodic orbits and KAM tori in the standard map. The periodic orbits have a rational winding number while the KAM tori have an irrational winding number. A periodic orbit with winding number $w(p_0) = \frac{N}{M}$ is called an M-cycle and has the property that $x_M = x_0^{(M)} + N \pmod 1$ and $p_M = p_0^{(M)}$, where $(p_0^{(M)}, x_0^{(M)})$ denote the coordinates of one member of the M-cycle.

Invariant Curves

K ≈ *0*

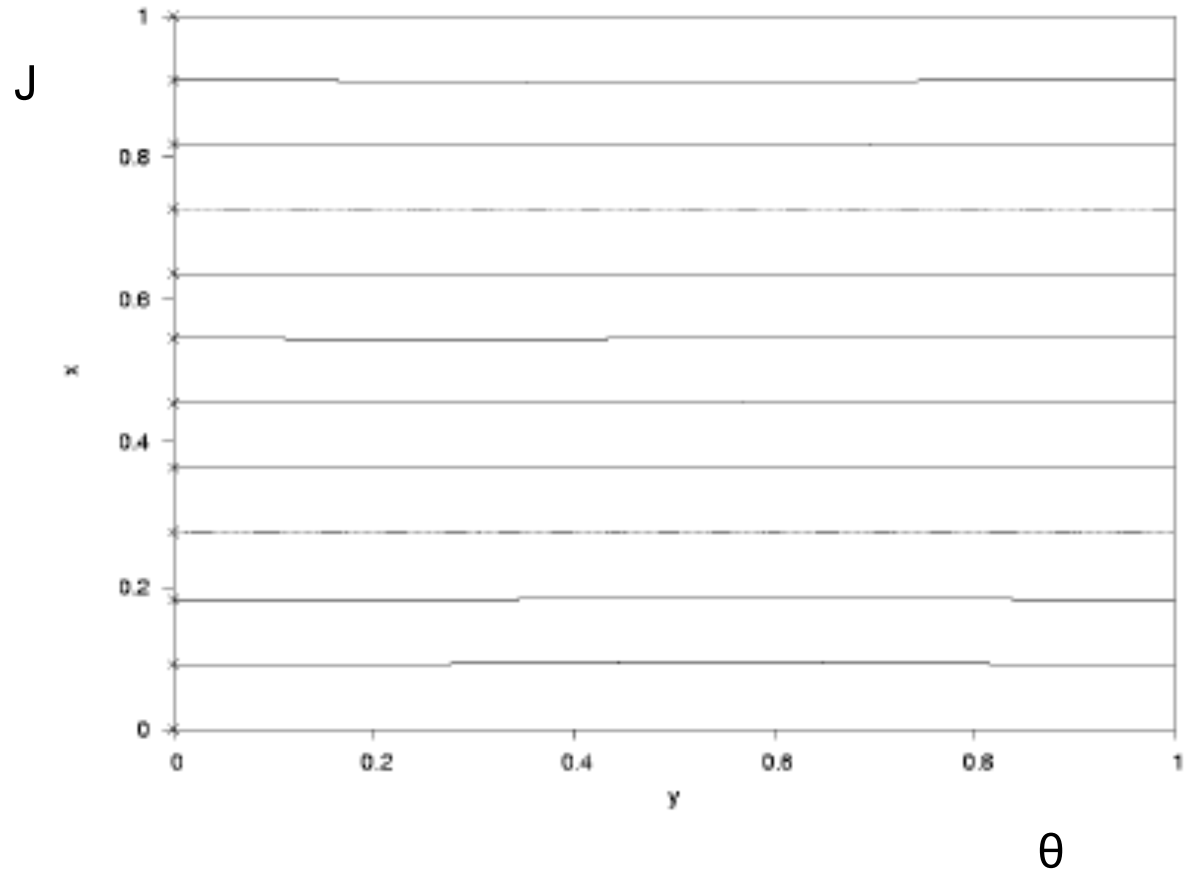


Figura 4.1: Mapa Padrão, $K = 10^{-3}$.

the M-cycles will be either elliptic or hyperbolic. For the standard map, these periodic orbits are particularly easy to find numerically because of a symmetry property [Greene 1979a]. The standard map, T_K, can be written as a product of two involutions, I_1 and I_2, such that

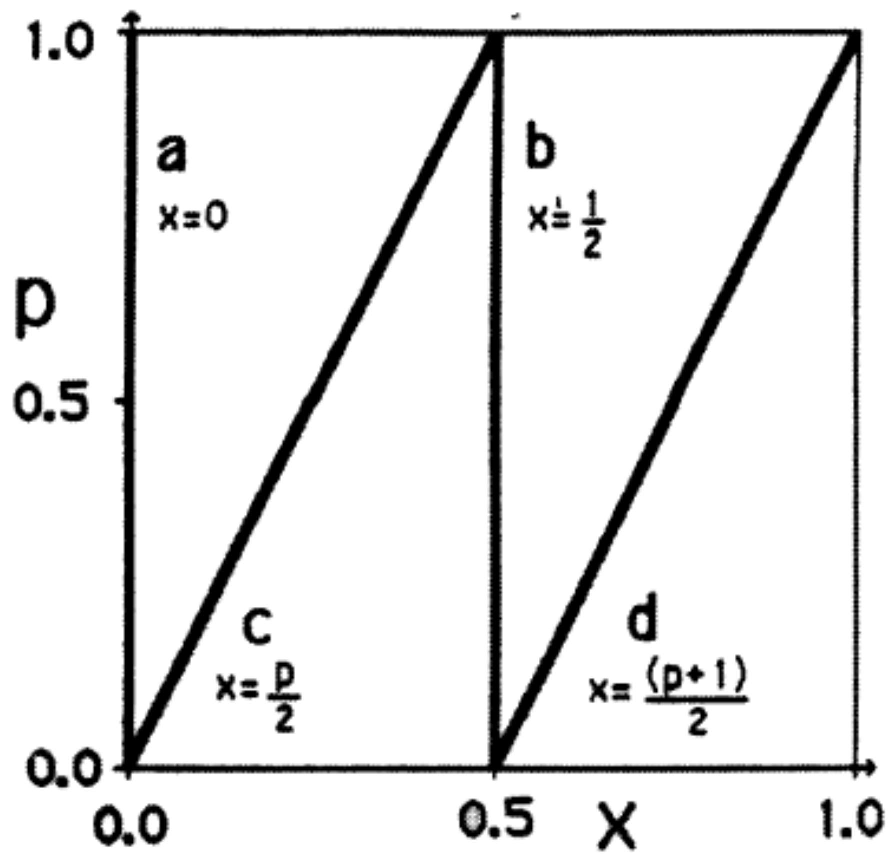
$$T_K = I_2 I_1,$$

$$I_1 \begin{pmatrix} p \\ x \end{pmatrix} = \begin{pmatrix} p - \frac{K}{2\pi}\sin(2\pi x) \\ -x \end{pmatrix}$$

$$I_2 \begin{pmatrix} p \\ x \end{pmatrix} = \begin{pmatrix} p \\ p - x \end{pmatrix}$$

$$I_1^2 = I_2^2 = \text{identity map, and } \det I_1 = \det I_2 = -1$$

Each of these involutions has lines of fixed points; that is, lines of points for which $I_1\begin{pmatrix} p \\ x \end{pmatrix} = \begin{pmatrix} p \\ x \end{pmatrix}$ and $I_2\begin{pmatrix} p \\ x \end{pmatrix} = \begin{pmatrix} p \\ x \end{pmatrix}$. For I_1, the lines of fixed points are $x = 0$ and $x = \frac{1}{2}$, while, for I_2, $x = \frac{p}{2}$ and $x = \frac{p+1}{2}$ are lines of fixed points.



Pontos fixos:
(x=0, p=0)
(x=0.5 , p=0)

Standard Map

$$p_{n+1} = p_n - \frac{k\ sen(2\pi x_n)}{2\pi}$$

$$x_{n+1} = x_n + p_{n+1}$$

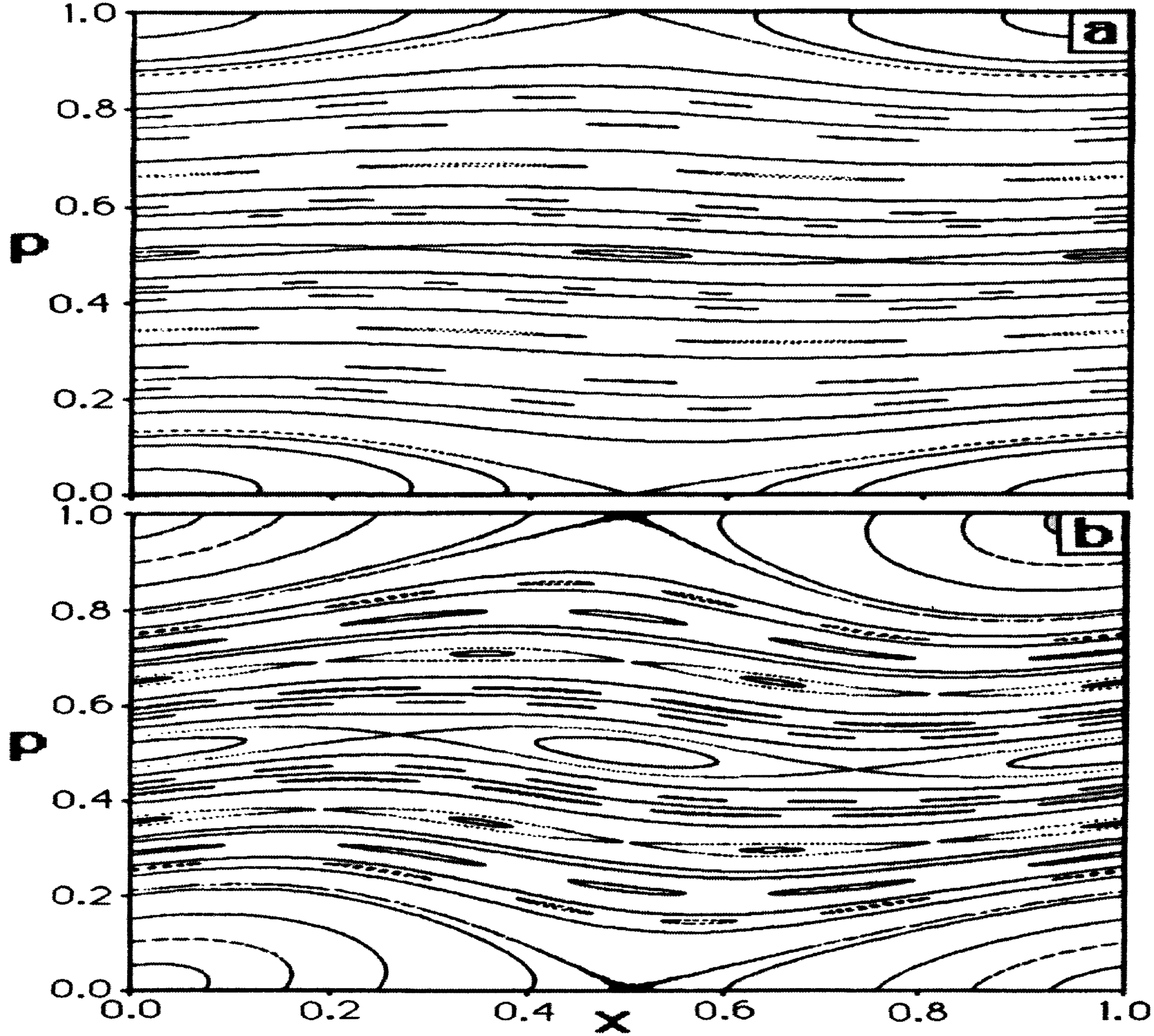
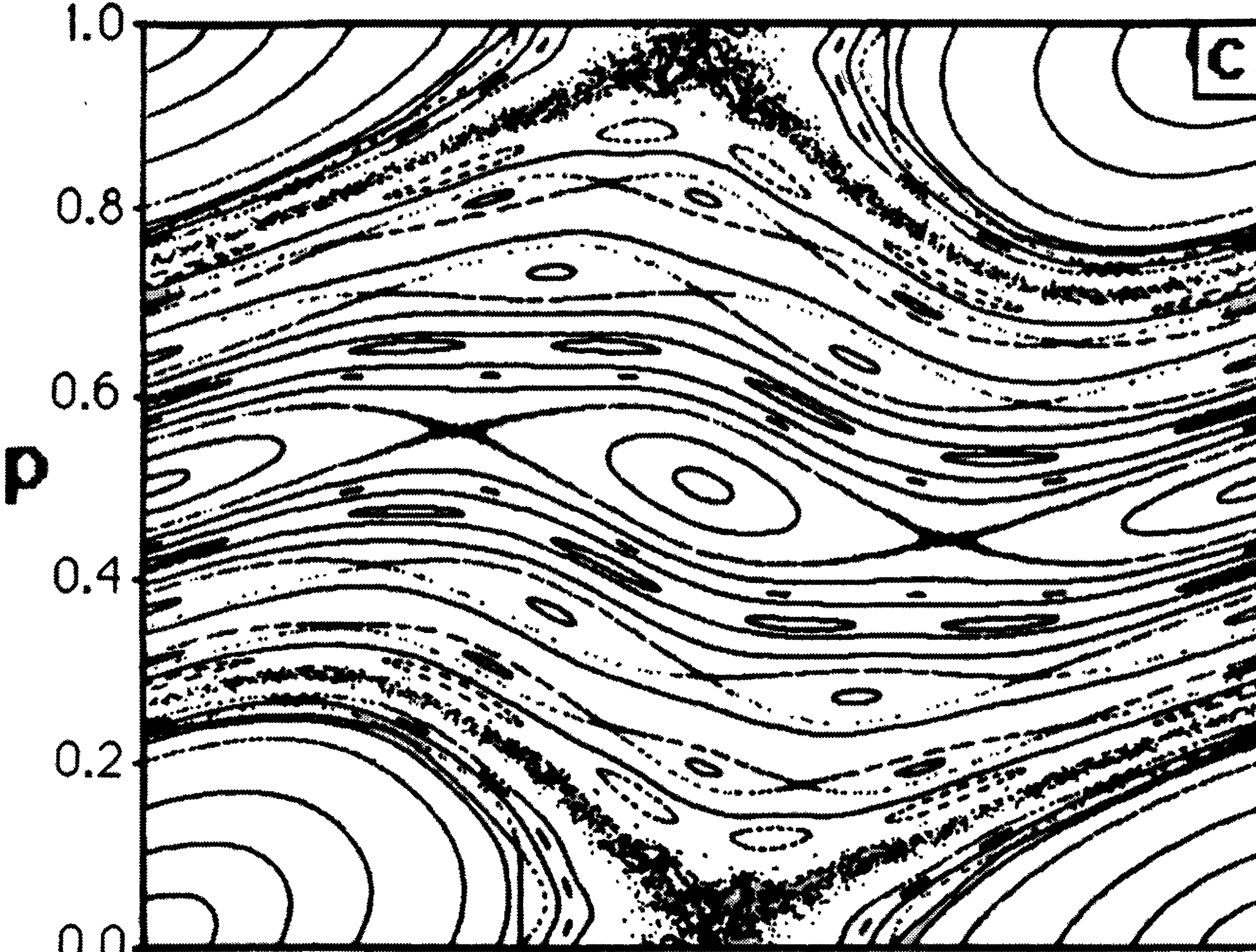


Figure 3.5.3. Some orbits of the standard map (with periodic boundary conditions): (a) $K = 0.1716354$; (b) $K = 0.4716354$.

Mudança em x

K = 0.7716354



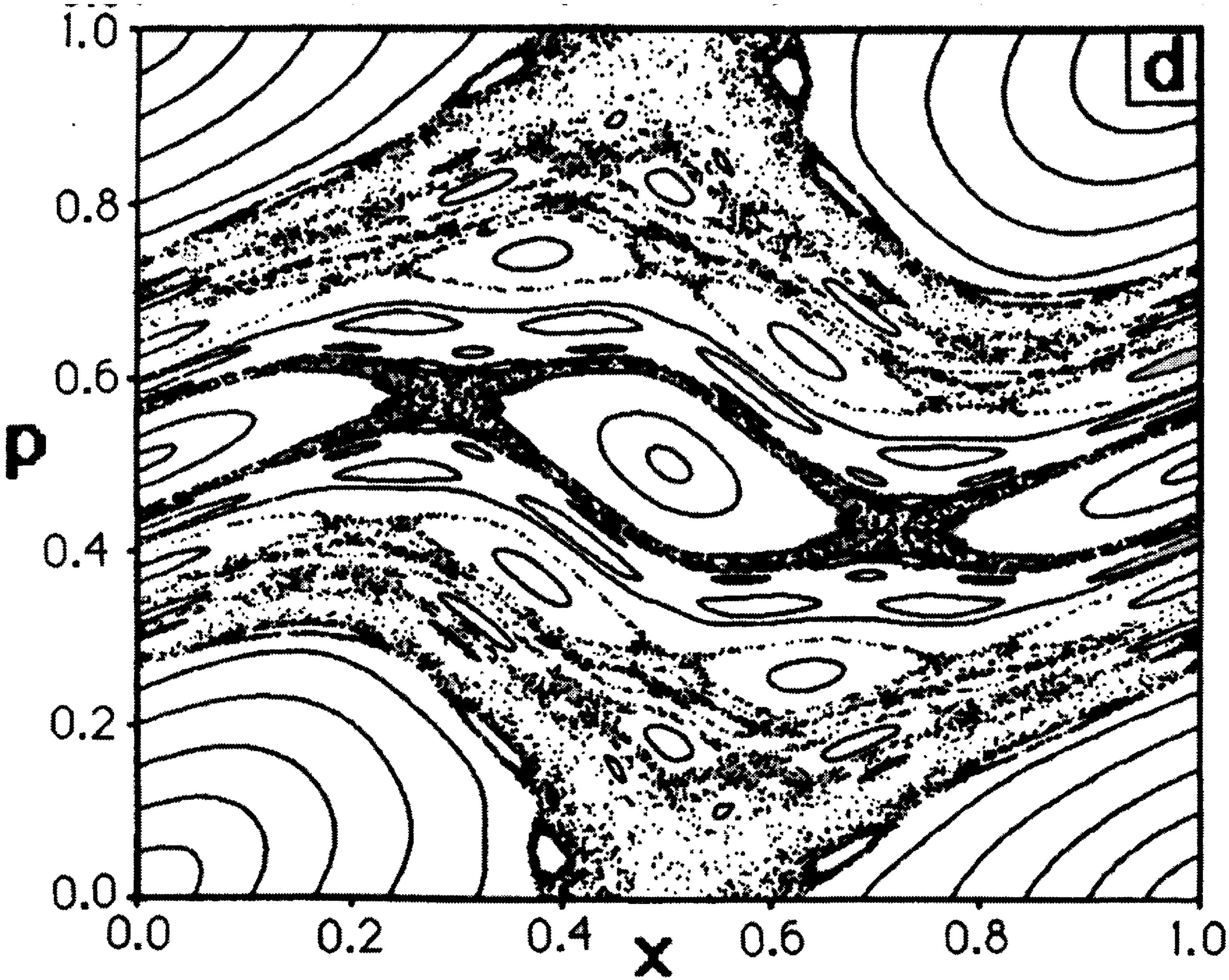
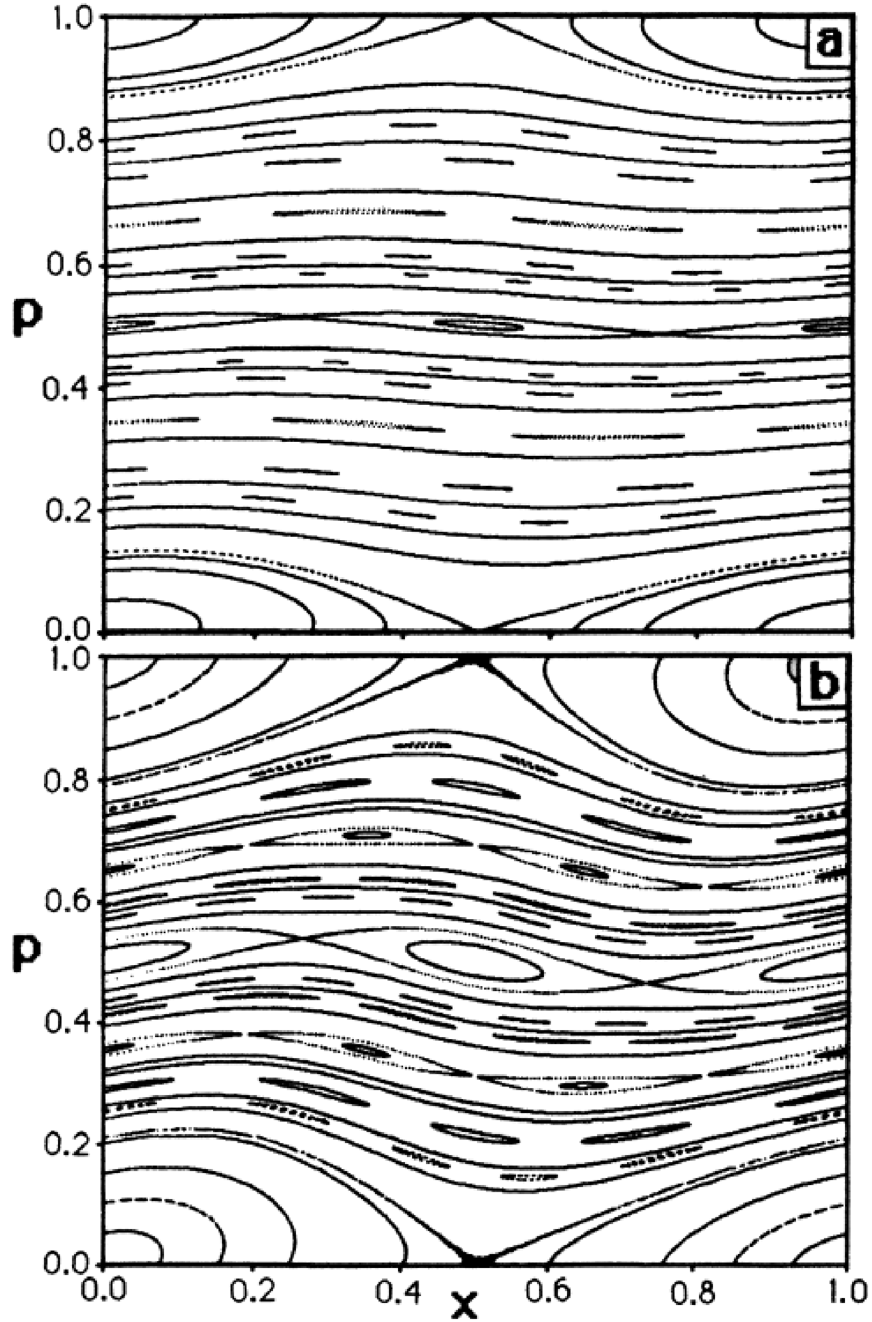


Figure 3.5.3. (*continued*) (c) $K = 0.7716354$; (d) $K = K^* = 0.9716354$.



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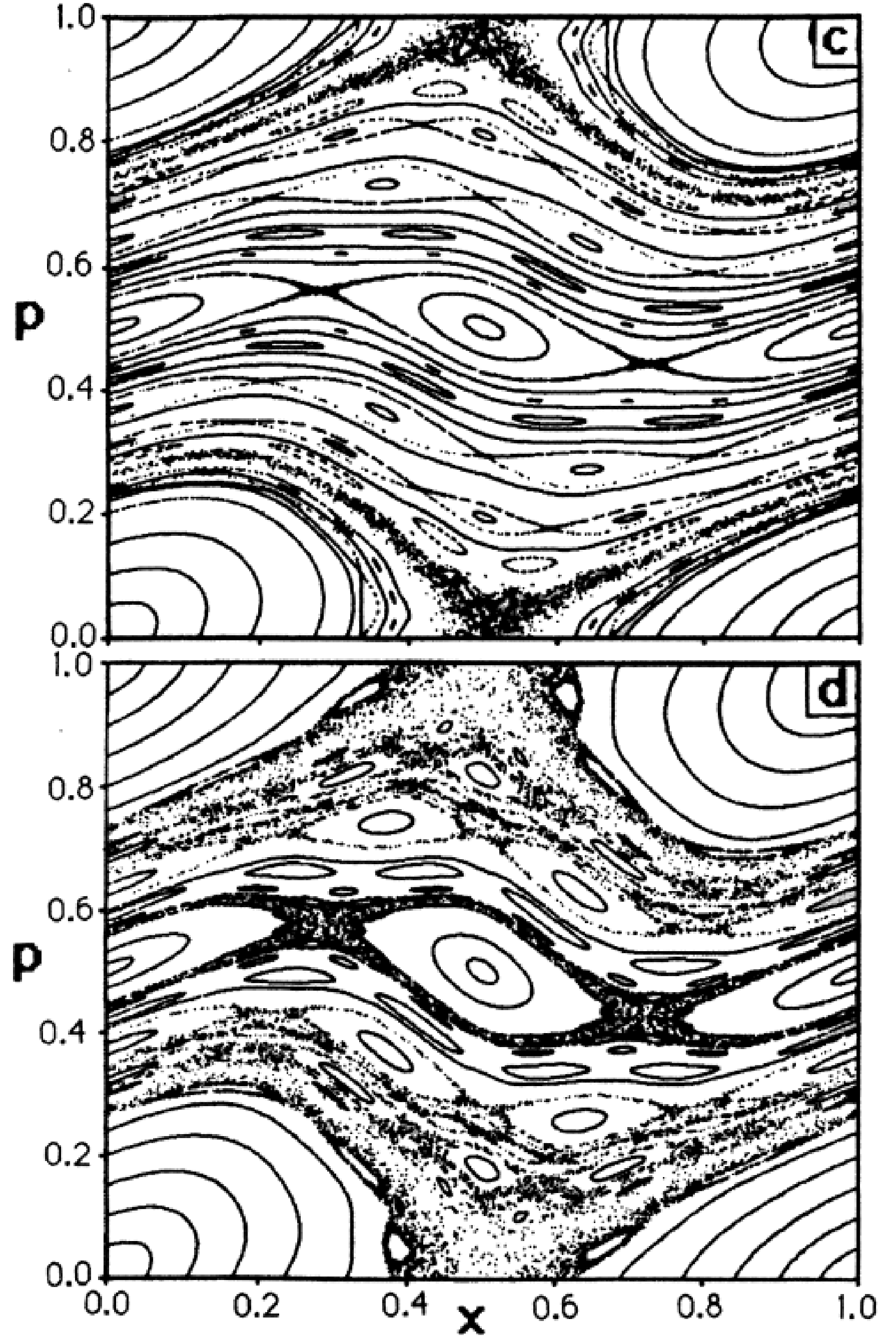

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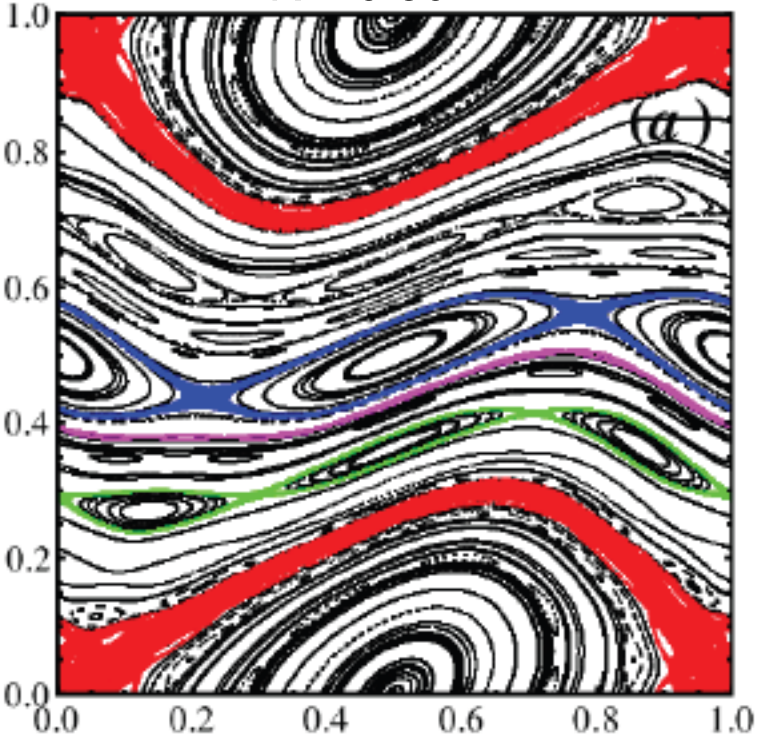
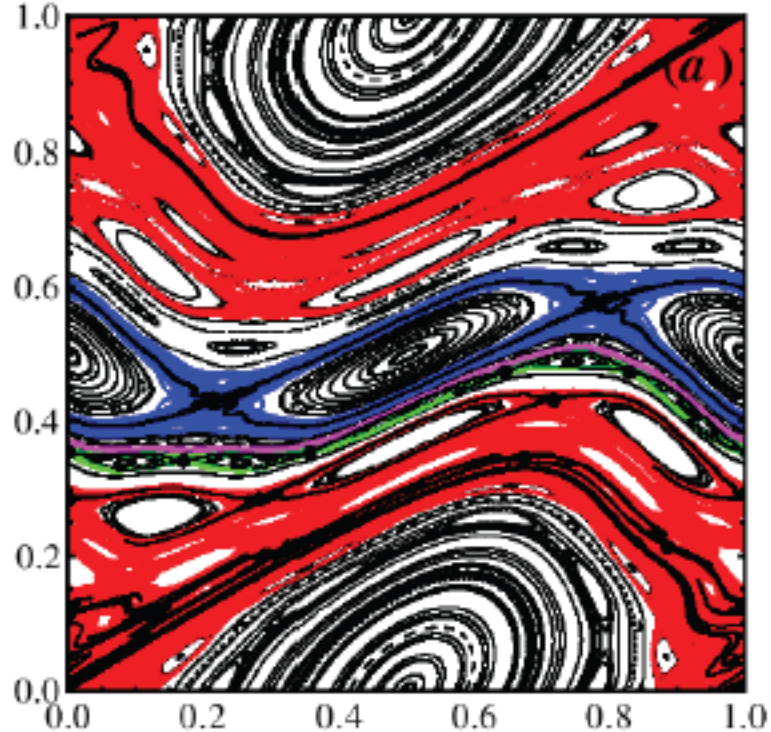
Caos Confinado / Barreiras KAM

K = 0.80



K = 0.95



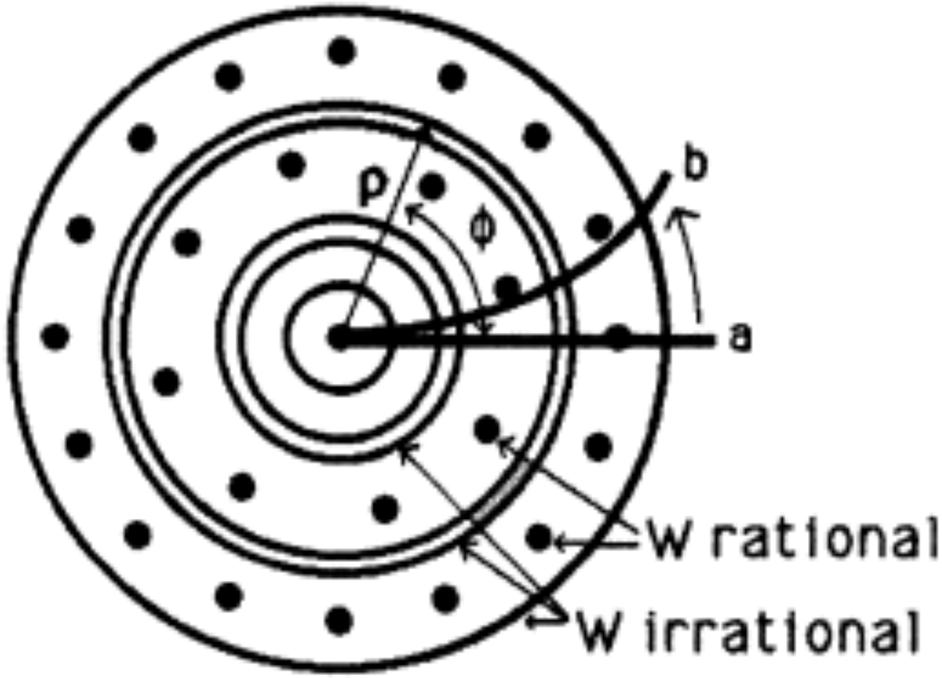
Cores diferentes indicam domínios caóticos confinados

arXiv:1810.1129v1, M. Harsoula, K. Karamanos, G. Contopoulos (2018)

III – Estabilidade de Pontos Fixos

(Baseado na seção *3. Area-Preserving Maps* do livro *The Transition to Chaos* de L. Reichl)

Estabilidade de pontos peródicos



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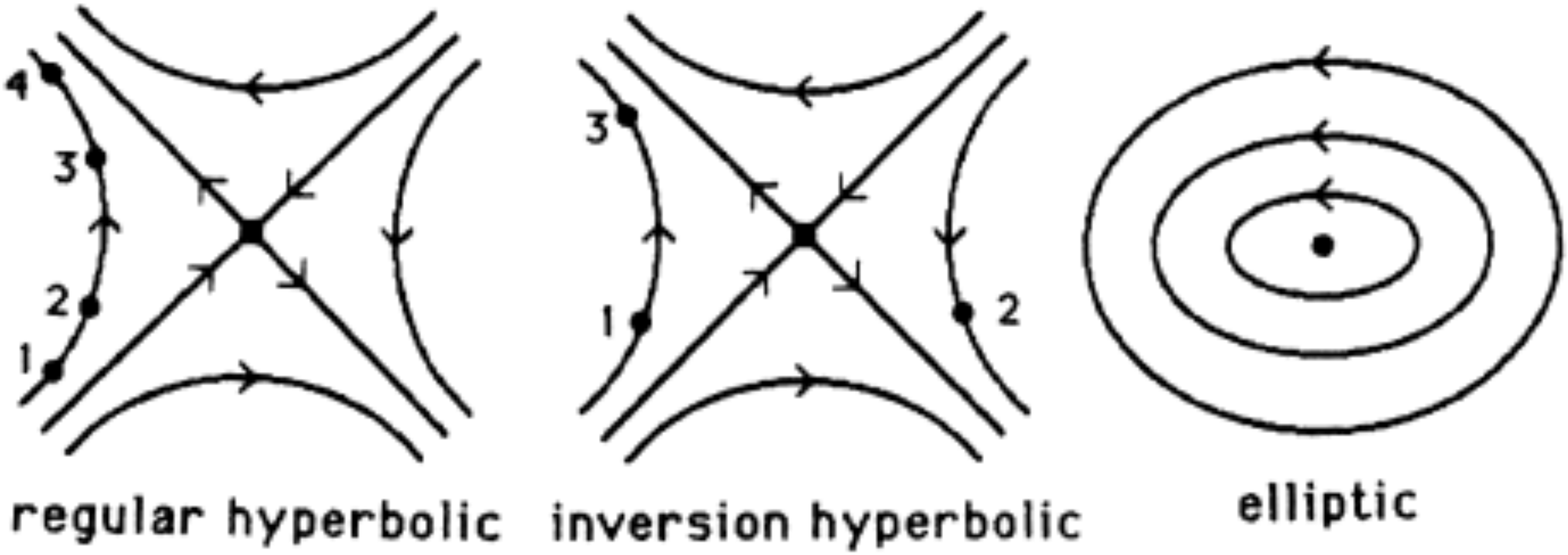

Figure 3.2.4. The flow of points in the neighborhood of fixed points. For regular hyperbolic points (residue $R < 0$), successive points on an orbit remain on one side of the fixed point, while for inversion hyperbolic points (residue $R > 1$) successive points alternate across the fixed point. The numbers indicate the sequence in time of the points. (The residue, R, is defined in Sect. 3.5.)

3.2.4 The Tangent Map

If we know the location of a given fixed point $X^{(0)} = \begin{pmatrix} \rho^{(0)} \\ \phi^{(0)} \end{pmatrix}$, where $X^{(0)} = T_\epsilon^M X^{(0)}$, we can determine its character by linearizing the mapping, T_ϵ^M, about the fixed point. The linearized mapping, ∇T_ϵ^M, is called the *tangent map*. Its eigenvalues are sometimes called the "multipliers" of the fixed point. To obtain ∇T_ϵ^M, linearize $X_n = \begin{pmatrix} \rho_n \\ \phi_n \end{pmatrix}$ about the fixed point $X^{(0)}$. That is, let $X_n = X^{(0)} + \delta X_n$, where δX_n is small. Then

$$\delta X_{n+1} = \nabla T_\epsilon^M \delta X_n, \qquad (3.2.11)$$

where

$$\nabla T_\epsilon^M = \begin{pmatrix} \frac{\partial \rho_{n+1}}{\partial \rho_n} & \frac{\partial \rho_{n+1}}{\partial \phi_n} \\ \frac{\partial \phi_{n+1}}{\partial \rho_n} & \frac{\partial \phi_{n+1}}{\partial \phi_n} \end{pmatrix}_{X^{(0)}}. \qquad (3.2.12)$$

We can determine the character of a particular M-cycle by linearizing the standard map in Eq. (3.5.1) about the coordinates of that M-cycle. After M steps, an initial point, $(\tilde{x}_0, \tilde{p}_0)$, on an M-cycle gets mapped to point $(\tilde{x}_M, \tilde{p}_M)$ via the mapping

$$\begin{pmatrix} \tilde{p}_M \\ \tilde{x}_M \end{pmatrix} = T_K^M \begin{pmatrix} \tilde{p}_0 \\ \tilde{x}_0 \end{pmatrix}, \tag{3.5.6}$$

where $\tilde{p}_M = \tilde{p}_0$ and $\tilde{x}_M = \tilde{x}_0 + N (\bmod 1) = \tilde{x}_0$. Let us now linearize this mapping in the neighborhood of the M-cycle. That is, we let $x_n = \tilde{x}_0 + \delta x_n$ and $p_n = \tilde{p}_0 + \delta p_n$. Then we can write

$$\begin{pmatrix} \delta p_M \\ \delta x_M \end{pmatrix} = \nabla T_K^M \begin{pmatrix} \delta p_0 \\ \delta x_0 \end{pmatrix}, \tag{3.5.7}$$

where

$$\nabla T_K^M = \begin{pmatrix} \frac{\partial p_M}{\partial p_0} & \frac{\partial p_M}{\partial x_0} \\ \frac{\partial x_M}{\partial p_0} & \frac{\partial x_M}{\partial x_0} \end{pmatrix}_{(x_0 = \tilde{x}_0,\ p_0 = \tilde{p}_0)} . \tag{3.5.8}$$

The eigenvalues of ∇T_ϵ^M are given by

$$\lambda^2 - \lambda \operatorname{Tr}(\nabla T_\epsilon^M) + \det(\nabla T_\epsilon^M) = 0. \tag{3.2.13}$$

But for area-preserving maps, $\det(\nabla T_\epsilon^M) = 1$, so the eigenvalues are given by

$$\lambda_\pm = \frac{t}{2} \pm \sqrt{\frac{t^2}{4} - 1}, \tag{3.2.14}$$

where $t = \operatorname{Tr}(\nabla T_\epsilon^M)$. Thus, the eigenvalues come in reciprocal pairs, $\lambda_+ = \lambda_-^{-1}$. For $-2 < t < 2$, the eigenvalues form complex conjugate pairs that lie on the unit circle, and the fixed points are elliptic. For $t > 2$, the fixed point is regular hyperbolic. For $t < -2$, the fixed point is inversion hyperbolic (subsequent points of the mapping alternate across the fixed point (see Fig. 3.2.4). For the special cases $t = \pm 2$, the eigenvalues are degenerate, having values $+1$ or -1, and the fixed point is parabolic. Parabolic fixed points are generally unstable [MacKay 1982].

Since the standard map is area-preserving, $\mathrm{Det}(\nabla T_K^M) = \lambda_1 \lambda_2 = 1$. If $\lambda_1 = \lambda_2{}^*$, the M-cycle is elliptic. If $\lambda_1 = \frac{1}{\lambda_2}$ (λ real), the M-cycle is hyperbolic. From our results in Sect. 3.2, the standard map will have two M-cycles, one elliptic and one hyperbolic (as long as K is small enough), for each value of M and N relatively prime.

If the mapping is defined in terms of smooth continuous functions, the eigenvectors of ∇T_ϵ^M in the neighborhood of the fixed point will be smooth and continuous. For elliptic fixed points, the eigenvalues will be pure imaginary and the eigenvectors will describe motion that oscillates about the fixed point. For hyperbolic fixed points, the eigenvalues will be real and of the form $\lambda_1 = \frac{1}{\lambda}$ and $\lambda_2 = \lambda$, where λ is real and $\lambda > 1$. Let us denote the eigencurve associated with eigenvalue $\frac{1}{\lambda}$ as $W^{(s)}$ and the eigencurve associated with eigenvalue λ as $W^{(u)}$. Once the eigencurves of the tangent map have been found, they can be extended away from the neighborhood of the fixed point by using the full map, T_ϵ^M. These extensions of the eigencurves are also denoted $W^{(s)}$ and $W^{(u)}$, and are called the *stable manifold* and *unstable manifold*, respectively. Points on $W^{(s)}$ will be mapped toward the fixed point since $(\nabla T_\epsilon^M)^n W^{(s)} = (\frac{1}{\lambda})^n W^{(s)}$, while points on $W^{(u)}$ will be mapped away from the fixed point since $(\nabla T_\epsilon^M)^n W^{(u)} = \lambda^n W^{(u)}$ (see Fig. 3.2.5).